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Eigenvector approach for solving nonisotropic mixing formulas

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Abstract. This paper deals with mixtures of nonisotropic media. The mixing formulas to describe the effective material parameters of the mixture are complicated expressions where the different parameter dyadics are coupled. Using six-vector notation, the different mixing rules become more manageable. However, for certain effective theories the matrix equations have been tractable only through numerical approach. Here we present a way to solve the effective material parameters in closed form, using an approach where the eigenvectors of the material six matrix are exploited. As a numerical example, the effective parameters of a mixture are calculated where the inclusion material is a mirror image of the background medium.

1. Introduction

The family of mixing formulas according to [*Sihvola*, 1989]

$$\frac{\varepsilon_{\text{eff}} - \varepsilon_0}{\varepsilon_{\text{eff}} + 2\varepsilon_0 + \nu(\varepsilon_{\text{eff}} - \varepsilon_0)} = f \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0 + \nu(\varepsilon_{\text{eff}} - \varepsilon_0)} \tag{1}$$

predicts the effective permittivity ε_{eff} of a mixture where inclusions of permittivity ε are embedded in free-space background (ε_0). The volume fraction of the inclusions is f. This expression contains important mixing rules that can be picked up with the dimensionless parameter ν. For example, the choice $\nu = 0$ reproduces the Maxwell Garnett formula [*Maxwell Garnett*, 1904]:

$$\varepsilon_{\text{eff}} = \varepsilon_0 + 3f\varepsilon_0 \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0 - f(\varepsilon - \varepsilon_0)} \tag{2}$$

Other integer values for ν give other well-known mixing rules. The value $\nu = 3$ gives the so-called "Coherent Potential (CP) formula" [*Kohler and Papanicolaou*, 1981]. Correspondingly, $\nu = 2$ gives the Böttcher mixing rule [*Bruggeman*, 1935; *Böttcher*, 1952], often labeled as the "Bruggeman formula."

To look at more complex materials, bianisotropic media obey the constitutive relations

$$\bar{D} = \bar{\bar{\varepsilon}} \cdot \bar{E} + \bar{\bar{\xi}} \cdot \bar{H} \tag{3}$$

$$\bar{B} = \bar{\bar{\zeta}} \cdot \bar{E} + \bar{\bar{\mu}} \cdot \bar{H} \tag{4}$$

Here the relation between the electric and magnetic fields ($\bar{E}$ and $\bar{H}$) and the electric and magnetic flux densities ($\bar{D}$ and $\bar{B}$) is contained in the permittivity $\bar{\bar{\varepsilon}} = \varepsilon_0 \bar{\bar{\varepsilon}}_r$ and permeability $\bar{\bar{\mu}} = \mu_0 \bar{\bar{\mu}}_r$ and the magnetoelectric crosscoupling in $\bar{\bar{\xi}}$ and $\bar{\bar{\zeta}}$. The dyadic nature of these material quantities emphasizes the anisotropy of the material. The biisotropic material is an important special case [*Lindell et al.*, 1994] which requires four parameters: All dyadics are (because of isotropy) multiples of the unit dyadic: $\bar{\bar{\varepsilon}} = \varepsilon \bar{\bar{I}}$, $\bar{\bar{\mu}} = \mu \bar{\bar{I}}$, $\bar{\bar{\xi}} = (\chi - j\kappa)\sqrt{\mu_0 \varepsilon_0}\, \bar{\bar{I}}$, and $\bar{\bar{\zeta}} = (\chi + j\kappa)\sqrt{\mu_0 \varepsilon_0}\, \bar{\bar{I}}$. Here χ and κ are the dimensionless nonreciprocity and chirality parameters, respectively, and μ_0, ε_0 are the vacuum constants.

Six-vector approach has been developed to account for bianisotropic effects in electromagnetics problems [*Lindell et al.*, 1995]. Six vectors combine electric and magnetic quantities (that both have three components) into a single vector with six components. The electromagnetic six-vector field **e** and six-vector flux density **d** look like

$$\mathbf{e} = \begin{bmatrix} \bar{E} \\ \eta \bar{H} \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} c\eta \bar{D} \\ c\bar{B} \end{bmatrix} \tag{5}$$

where use is made of the idea by *Rikte* [1994] to balance dimensionally the fields and flux densities. The constants are $c = 1/\sqrt{\mu_0 \varepsilon_0}$ and $\eta = \sqrt{\mu_0/\varepsilon_0}$. Then the constitutive relations (3)–(4) can be written as a single equation:

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$$\mathbf{d} = \mathbf{M} \cdot \mathbf{e} \tag{6}$$

where

$$\mathbf{M} = \begin{bmatrix} \bar{\bar{\varepsilon}}_r & c\bar{\bar{\xi}} \\ c\bar{\bar{\zeta}} & \bar{\bar{\mu}}_r \end{bmatrix} \tag{7}$$

is the six dyadic of the material parameters. It has a 6×6 element matrix representation, and the full medium description requires 36 parameters.

Using six vectors and six dyadics, the dielectric mixing rule (1) can be generalized into mixtures with bianisotropic inclusions. *Sihvola and Pekonen* [1996] make the conjecture that the effective material six dyadic

$$\mathbf{M}_{\text{eff}} = \begin{bmatrix} \bar{\bar{\varepsilon}}_{\text{eff,r}} & c\bar{\bar{\xi}}_{\text{eff}} \\ c\bar{\bar{\zeta}}_{\text{eff}} & \bar{\bar{\mu}}_{\text{eff,r}} \end{bmatrix} \tag{8}$$

obeys the formula

$$(\mathbf{M}_{\text{eff}} - \mathbf{M}_0) \cdot [\mathbf{M}_{\text{eff}} + 2\mathbf{M}_0 + \nu(\mathbf{M}_{\text{eff}} - \mathbf{M}_0)]^{-1}$$
$$= f(\mathbf{M} - \mathbf{M}_0) \cdot [\mathbf{M} + 2\mathbf{M}_0 + \nu(\mathbf{M}_{\text{eff}} - \mathbf{M}_0)]^{-1} \tag{9}$$

The formula (9) satisfies the tests of reducing to all previously known isotropic and nonisotropic mixing rules. The inclusion six-dyadic $\mathbf{M}$ can be fully bianisotropic (36 parameters), and the medium on which the polarizabilities are referred to is allowed to be biisotropic (four parameters) in the most general case. This medium is, for example, $\mathbf{M}_0$ for Maxwell Garnett and $\mathbf{M}_{\text{eff}}$ for the Bruggeman case. Note, however, that since the terms in (9) are dyadic, the order of multiplication and inversions needs more attention than in the scalar isotropic case. Equation (9) is a second-order matrix equation for $\mathbf{M}_{\text{eff}}$ which is more difficult to solve than the pure scalar dielectric case (1). The iterative approach to (9) has been shown to be successful [*Sihvola and Pekonen*, 1996]. In the following we look for a direct approach to derive $\mathbf{M}_{\text{eff}}$.

2. Eigenvector Solution

We now will derive a systematic solution of the matrix (9). In general, it is possible to diagonalize every matrix except for very special cases where the algebraic and geometric multiplicity of the eigenvalues differ. We will not consider these special cases here. As a first step, we diagonalize the matrix $\mathbf{M}_0$ as

$$\mathbf{M}_0 = \mathbf{V}_0 \cdot \mathbf{L}_0 \cdot \mathbf{V}_0^{-1} \tag{10}$$

where $\mathbf{L}_0$ is a diagonal matrix with the eigenvalues of the matrix $\mathbf{M}_0$ on its diagonal and $\mathbf{V}_0$ is a matrix with the eigenvectors of $\mathbf{M}_0$ as its columns. (Although the background material is not allowed to be fully bianisotropic, it is not necessarily diagonal. For isotropic chiral and nonreciprocal media there are off-diagonal terms in the matrix $\mathbf{M}_0$.) From the diagonalization of $\mathbf{M}_0$ it is easy to construct its square root denoted by $\sqrt{\mathbf{M}_0}$:

$$\sqrt{\mathbf{M}_0} = \mathbf{V}_0 \cdot \sqrt{\mathbf{L}_0} \cdot \mathbf{V}_0^{-1} \tag{11}$$

where the square root of the diagonal matrix is the diagonal matrix with the square roots of the diagonal elements. After multiplication from the left by $\sqrt{\mathbf{M}_0}^{-1}$ and from the right by $\sqrt{\mathbf{M}_0}$, (9) can be rewritten as

$$\sqrt{\mathbf{M}_0}^{-1} \cdot (\mathbf{M}_{\text{eff}} - \mathbf{M}_0) \cdot \sqrt{\mathbf{M}_0}^{-1} \cdot \sqrt{\mathbf{M}_0} \cdot [\mathbf{M}_{\text{eff}} + 2\mathbf{M}_0$$
$$+ \nu(\mathbf{M}_{\text{eff}} - \mathbf{M}_0)]^{-1} \cdot \sqrt{\mathbf{M}_0} = f\sqrt{\mathbf{M}_0}^{-1}$$
$$\cdot (\mathbf{M} - \mathbf{M}_0) \cdot \sqrt{\mathbf{M}_0}^{-1} \cdot \sqrt{\mathbf{M}_0} \cdot [\mathbf{M} + 2\mathbf{M}_0$$
$$+ \nu(\mathbf{M}_{\text{eff}} - \mathbf{M}_0)]^{-1} \cdot \sqrt{\mathbf{M}_0} \tag{12}$$

Let us now define $\mathbf{M}_{\text{r}}$ and $\mathbf{M}_{\text{eff,r}}$ as

$$\mathbf{M}_{\text{r}} = \sqrt{\mathbf{M}_0}^{-1} \cdot \mathbf{M} \cdot \sqrt{\mathbf{M}_0}^{-1} \tag{13}$$

and

$$\mathbf{M}_{\text{eff,r}} = \sqrt{\mathbf{M}_0}^{-1} \cdot \mathbf{M}_{\text{eff}} \cdot \sqrt{\mathbf{M}_0}^{-1} \tag{14}$$

With these definitions, (12) becomes

$$(\mathbf{M}_{\text{eff,r}} - \mathbf{I}) \cdot [\mathbf{M}_{\text{eff,r}} + 2\mathbf{I} + \nu(\mathbf{M}_{\text{eff,r}} - \mathbf{I})]^{-1}$$
$$= f(\mathbf{M}_{\text{r}} - \mathbf{I}) \cdot [\mathbf{M}_{\text{r}} + 2\mathbf{I} + \nu(\mathbf{M}_{\text{eff,r}} - \mathbf{I})]^{-1} \tag{15}$$

where $\mathbf{I}$ is the 6×6 unit matrix. Multiplying from the right by $[\mathbf{M}_{\text{r}} + 2\mathbf{I} + \nu(\mathbf{M}_{\text{eff,r}} - \mathbf{I})]$ and from the left by $[\mathbf{M}_{\text{eff,r}} + 2\mathbf{I} + \nu(\mathbf{M}_{\text{eff,r}} - \mathbf{I})]$ and noting that the dyadics $\mathbf{M}_{\text{eff,r}} - \mathbf{I}$ and $\mathbf{M}_{\text{eff,r}} + 2\mathbf{I} + \nu(\mathbf{M}_{\text{eff,r}} - \mathbf{I})$ commute, we arrive at

$$\nu\mathbf{M}_{\text{eff,r}}^2 + \mathbf{M}_{\text{eff,r}} \cdot [\mathbf{M}_{\text{r}} + 2(1-\nu)\mathbf{I} - f(1+\nu)(\mathbf{M}_{\text{r}} - \mathbf{I})]$$
$$- [\mathbf{M}_{\text{r}} + (2-\nu)\mathbf{I} + f(2-\nu)(\mathbf{M}_{\text{r}} - \mathbf{I})] = 0 \tag{16}$$

(Two dyadics of the form $\alpha\mathbf{M}_{\text{eff,r}} + \beta\mathbf{I}$ with arbitrary scalar coefficients α and β commute.) To solve this equation, we also diagonalize the matrix $\mathbf{M}_{\text{r}}$ as

$$\mathbf{M}_{\text{r}} = \mathbf{V}_{\text{r}} \cdot \mathbf{L}_{\text{r}} \cdot \mathbf{V}_{\text{r}}^{-1} \tag{17}$$

where now $\mathbf{V}_r$ contains the eigenvectors of $\mathbf{M}_r$ and $\mathbf{L}_r$ its eigenvalues. Define a new matrix $\mathbf{L}_{\text{eff},r}$ as

$$\mathbf{L}_{\text{eff},r} = \mathbf{V}_r^{-1} \cdot \mathbf{M}_{\text{eff},r} \cdot \mathbf{V}_r \tag{18}$$

where as for now, the matrix $\mathbf{L}_{\text{eff},r}$ is of unknown type. With these definitions it is possible to rewrite (16) as

$$\nu \mathbf{L}_{\text{eff},r}^2 + \mathbf{L}_{\text{eff},r} \cdot [\mathbf{L}_r + 2(1-\nu)\mathbf{I} - f(1+\nu)(\mathbf{L}_r - \mathbf{I})]$$
$$- [\mathbf{L}_r + (2-\nu)\mathbf{I} + f(2-\nu)(\mathbf{L}_r - \mathbf{I})] = 0 \tag{19}$$

Since we know the matrices $\mathbf{L}_r$ and $\mathbf{I}$ to be diagonal, the unknown matrix $\mathbf{L}_{\text{eff},r}$ must also be a diagonal matrix. Consequently, the matrix equation is reduced to an uncoupled set of six scalar quadratic equations:

$$\nu \lambda_{\text{eff},r,i}^2 + \lambda_{\text{eff},r,i}[\lambda_{r,i} + 2(1-\nu) - f(\lambda_{r,i} - 1)(1+\nu)]$$
$$- [\lambda_{r,i} + 2 - \nu + f(\lambda_{r,i} - 1)(2-\nu)] = 0 \tag{20}$$

where $\lambda_{\text{eff},r,i}$ and $\lambda_{r,i}$, $(i = 1, \cdots, 6)$, are the diagonal elements of $\mathbf{L}_{\text{eff},r}$ and $\mathbf{L}_r$, respectively. The solution of (20) is given by

$$\lambda_{\text{eff},r,i} = -\frac{\lambda_{r,i} + 2(1-\nu) - f(\lambda_{r,i} - 1)(1+\nu) - \sqrt{\Delta}}{2\nu} \tag{21}$$

with

$$\Delta = f^2(\lambda_{r,i} - 1)^2(1+\nu)^2 - 2f(\lambda_{r,i} - 1)[\lambda_{r,i}(1+\nu)$$
$$+ 2(1-2\nu)] + [\lambda_{r,i} + 2(1-\nu)]^2$$
$$+ 4\nu(\lambda_{r,i} + 2 - \nu) \tag{22}$$

where we only kept the positive square root. In tracing back on our steps we can easily express $\mathbf{M}_{\text{eff}}$ as

$$\mathbf{M}_{\text{eff}} = \sqrt{\mathbf{M}_0} \cdot \mathbf{V}_r \cdot \mathbf{L}_{\text{eff},r} \cdot \mathbf{V}_r^{-1} \cdot \sqrt{\mathbf{M}_0} \tag{23}$$

which is the desired solution.

One may note that for isotropic background the determination of $\sqrt{\mathbf{M}_0}$ becomes trivial. In the case of fully biisotropic background medium the calculation of $\sqrt{\mathbf{M}_0}$ and also $\mathbf{M}_r$ for general bianisotropic inclusions can be done in closed form as shown in the appendix. Remark that the only difficulty in this procedure is the determination of the eigenvalues of $\mathbf{M}_r$. In general, this needs to be done numerically. Fortunately, very good numerical software packages are available to determine eigenvalues and eigenvectors, certainly for these low-dimensional matrices that we encounter here. Finally, we want to remark that these diagonalization procedures are independent of the volume fraction f and the nature of the mixing rule ν. (The appearance of ν in the denominator of (21) may bother those who are interested in looking for Maxwell Garnett results of effective media ($\nu = 0$ gives the Maxwell Garnett formula). However, for the MG case the starting equation itself (9) can be written in a linear explicit form for $\mathbf{M}_{\text{eff}}$ and calculated easily, and there is no need for the eigenvector approach at all.)

We intentionally used the subscript r in $\mathbf{M}_{\text{eff},r}$ and $\mathbf{M}_r$ to indicate that these are the material parameters relative to the background. In the case of a vacuum background medium the elements in the $\mathbf{M}_{\text{eff},r}$ and $\mathbf{M}_r$ matrices are the traditional relative material parameters.

3. Special Cases

In this section three examples will be studied to illustrate the solution technique. We will start with isotropic chiral inclusions embedded in isotropic background. In the second example the background will also become chiral. As mentioned, it is possible to construct $\mathbf{M}_{\text{eff}}$ in closed form as soon as the eigenvalues of the $\mathbf{M}_0$ and $\mathbf{M}_r$ matrices are known. For isotropic chiral media their calculation is simple. In the last special case, inclusion materials consisting of coaxial bianisotropic inclusions are considered.

3.1. Isotropic Chiral Inclusions in Vacuum Background

Vacuum background means $\mathbf{M}_0 = \mathbf{I}$, and (13) shows that $\mathbf{M}_r = \mathbf{M}$. For isotropic chiral inclusions the $\mathbf{M}_r$ matrix can be written as

$$\mathbf{M}_r = \begin{bmatrix} \varepsilon_r \bar{\bar{I}} & -j\kappa \bar{\bar{I}} \\ j\kappa \bar{\bar{I}} & \mu_r \bar{\bar{I}} \end{bmatrix} \tag{24}$$

The eigenvalues of this matrix are easily obtained. There are only two different eigenvalues $\lambda_{r,\pm}$ each with threefold multiplicity.

$$\lambda_{r,\pm} = \frac{\varepsilon_r + \mu_r}{2} \pm \sqrt{\left(\frac{\varepsilon_r - \mu_r}{2}\right)^2 + \kappa^2} \tag{25}$$

The eigenvector matrix $\mathbf{V}_r$ can be written with the two eigenvectors as the columns, and also its inverse is found easily:

$$\mathbf{V}_r = \begin{bmatrix} j\kappa \bar{\bar{I}} & j\kappa \bar{\bar{I}} \\ (\varepsilon_r - \lambda_{r,+})\bar{\bar{I}} & (\varepsilon_r - \lambda_{r,-})\bar{\bar{I}} \end{bmatrix} \tag{26}$$

$$\mathbf{V}_r^{-1} = \frac{1}{j\kappa(\lambda_{r,+} - \lambda_{r,-})} \begin{bmatrix} (\varepsilon_r - \lambda_{r,-})\bar{\bar{I}} & -j\kappa\bar{\bar{I}} \\ -(\varepsilon_r - \lambda_{r,+})\bar{\bar{I}} & j\kappa\bar{\bar{I}} \end{bmatrix} \tag{27}$$

The effective medium matrix comes from using (23) (note that $\mathbf{M}_0$ is the unit matrix now):

$$\mathbf{M}_{\text{eff}} = \mathbf{M}_{\text{eff,r}} = \frac{1}{\lambda_{r,+} - \lambda_{r,-}} \cdot \begin{bmatrix} [\lambda_{\text{eff,r},+}(\varepsilon_r - \lambda_{r,-}) - \lambda_{\text{eff,r},-}(\varepsilon_r - \lambda_{r,+})]\bar{\bar{I}} & -j\kappa(\lambda_{\text{eff,r},+} - \lambda_{\text{eff,r},-})\bar{\bar{I}} \\ j\kappa(\lambda_{\text{eff,r},+} - \lambda_{\text{eff,r},-})\bar{\bar{I}} & [\lambda_{\text{eff,r},-}(\varepsilon_r - \lambda_{r,-}) - \lambda_{\text{eff,r},+}(\varepsilon_r - \lambda_{r,+})]\bar{\bar{I}} \end{bmatrix} \tag{28}$$

This is an explicit result for the effective material parameters of the chiral-in-vacuum mixture for all different mixing rules. Earlier in the literature these results have been calculated iteratively only [*Kampia and Lakhtakia*, 1992; *Sihvola and Pekonen*, 1996].

3.2. Isotropic Chiral Inclusions in Isotropic Chiral Background

Let us next allow also the background medium to be chiral. Denote the background parameter values with a tilde, whence $\mathbf{M}_0$ is of the form

$$\mathbf{M}_0 = \begin{bmatrix} \tilde{\varepsilon}_r\bar{\bar{I}} & -j\tilde{\kappa}\bar{\bar{I}} \\ j\tilde{\kappa}\bar{\bar{I}} & \tilde{\mu}_r\bar{\bar{I}} \end{bmatrix} \tag{29}$$

The two different eigenvalues are

$$\lambda_{0,\pm} = \frac{\tilde{\varepsilon}_r + \tilde{\mu}_r}{2} \pm \sqrt{\left(\frac{\tilde{\varepsilon}_r - \tilde{\mu}_r}{2}\right)^2 + \tilde{\kappa}^2} \tag{30}$$

This means that we can write $\sqrt{\mathbf{M}_0}^{-1}$ as

$$\sqrt{\mathbf{M}_0}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{\hat{\varepsilon}_r}}\bar{\bar{I}} & -j\frac{1}{\sqrt{\hat{\kappa}}}\bar{\bar{I}} \\ j\frac{1}{\sqrt{\hat{\kappa}}}\bar{\bar{I}} & \frac{1}{\sqrt{\hat{\mu}_r}}\bar{\bar{I}} \end{bmatrix} \tag{31}$$

with

$$\frac{1}{\sqrt{\hat{\varepsilon}_r}} = \frac{\tilde{\mu}_r + \sqrt{\lambda_{0,+}\lambda_{0,-}}}{\sqrt{\lambda_{0,+}\lambda_{0,-}}(\sqrt{\lambda_{0,+}} + \sqrt{\lambda_{0,-}})} \tag{32}$$

$$\frac{1}{\sqrt{\hat{\kappa}}} = \frac{-\tilde{\kappa}}{\sqrt{\lambda_{0,+}\lambda_{0,-}}(\sqrt{\lambda_{0,+}} + \sqrt{\lambda_{0,-}})} \tag{33}$$

$$\frac{1}{\sqrt{\hat{\mu}_r}} = \frac{\tilde{\varepsilon}_r + \sqrt{\lambda_{0,+}\lambda_{0,-}}}{\sqrt{\lambda_{0,+}\lambda_{0,-}}(\sqrt{\lambda_{0,+}} + \sqrt{\lambda_{0,-}})} \tag{34}$$

If the inclusions are described by the six-dyadic $\mathbf{M}$ of the form

$$\mathbf{M} = \begin{bmatrix} \varepsilon_r\bar{\bar{I}} & -j\kappa\bar{\bar{I}} \\ j\kappa\bar{\bar{I}} & \mu_r\bar{\bar{I}} \end{bmatrix} \tag{35}$$

then the matrix $\mathbf{M}_r$ is given by:

$$\mathbf{M}_r = \begin{bmatrix} \check{\varepsilon}_r\bar{\bar{I}} & -j\check{\kappa}\bar{\bar{I}} \\ j\check{\kappa}\bar{\bar{I}} & \check{\mu}_r\bar{\bar{I}} \end{bmatrix} \tag{36}$$

with

$$\check{\varepsilon}_r = \frac{\varepsilon_r}{\hat{\varepsilon}_r} + \frac{\mu_r}{\hat{\kappa}} + 2\frac{\kappa}{\sqrt{\hat{\kappa}\hat{\varepsilon}_r}} \tag{37}$$

$$\check{\mu}_r = \frac{\mu_r}{\hat{\mu}_r} + \frac{\varepsilon_r}{\hat{\kappa}} + 2\frac{\kappa}{\sqrt{\hat{\kappa}\hat{\mu}_r}} \tag{38}$$

$$\check{\kappa} = \frac{\varepsilon_r}{\sqrt{\hat{\kappa}\hat{\varepsilon}_r}} + \frac{\mu_r}{\sqrt{\hat{\kappa}\hat{\mu}_r}} + \frac{\kappa}{\hat{\kappa}} + \frac{\kappa}{\sqrt{\hat{\varepsilon}_r\hat{\mu}_r}} \tag{39}$$

Now the problem has been reset into the complexity of the previous section. To find $\mathbf{M}_{\text{eff,r}}$ one proceeds as was done for the case with vacuum background. The final step, calculating the absolute material parameters in $\mathbf{M}_{\text{eff}}$, is done easily by multiplying $\mathbf{M}_{\text{eff,r}}$ from the left and the right with $\sqrt{\mathbf{M}_0}$ as shown in (23).

3.3. Coaxial Bianisotropic Inclusions

As the last example, we look at a more complex inclusion material which still allows a closed form diagonalization of its six-dyadic $\mathbf{M}$. We assume that the four dyadics $\bar{\bar{\varepsilon}}_r$, $\bar{\bar{\mu}}_r$, $\bar{\bar{\zeta}}$, and $\bar{\bar{\xi}}$ are general diagonal matrices

$$\bar{\bar{\alpha}} = \alpha_x\mathbf{u}_x\mathbf{u}_x + \alpha_y\mathbf{u}_y\mathbf{u}_y + \alpha_z\mathbf{u}_z\mathbf{u}_z \tag{40}$$

with $\alpha = \varepsilon_r$, μ_r, ζ, and ξ. This material is said to be coaxial bianisotropic, sometimes also called biaxial or triaxial. From the appendix it follows for a general biisotropic background that in $\mathbf{M}_r$, given by

$$\mathbf{M}_r = \begin{bmatrix} \check{\bar{\bar{\varepsilon}}}_r & c\check{\bar{\bar{\xi}}} \\ c\check{\bar{\bar{\zeta}}} & \check{\bar{\bar{\mu}}}_r \end{bmatrix} \tag{41}$$

the dyadics $\check{\bar{\bar{\varepsilon}}}_r$, $\check{\bar{\bar{\xi}}}$, $\check{\bar{\bar{\zeta}}}$, and $\check{\bar{\bar{\mu}}}_r$ remain diagonal.

The six eigenvalues of $\mathbf{M}_r$ are given by

$$\lambda_{r,i,\pm} = \frac{\breve{\varepsilon}_{r,i} + \breve{\mu}_{r,i}}{2} \pm \sqrt{\left(\frac{\breve{\varepsilon}_{r,i} - \breve{\mu}_{r,i}}{2}\right)^2 + c^2 \breve{\xi}_i \breve{\zeta}_i} \qquad (42)$$

with $i = x$, y, and z. The eigenvector matrix $\mathbf{V}_r$ is found to be

$$\mathbf{V}_r = \begin{bmatrix} \lambda_{r,x,+} - \breve{\mu}_x & \lambda_{r,x,-} - \breve{\mu}_x & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_{r,y,+} - \breve{\mu}_y & \lambda_{r,y,-} - \breve{\mu}_y & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_{r,z,+} - \breve{\mu}_z & \lambda_{r,z,-} - \breve{\mu}_z \\ c\breve{\zeta}_x & c\breve{\zeta}_x & 0 & 0 & 0 & 0 \\ 0 & 0 & c\breve{\zeta}_y & c\breve{\zeta}_y & 0 & 0 \\ 0 & 0 & 0 & 0 & c\breve{\zeta}_z & c\breve{\zeta}_z \end{bmatrix} \qquad (43)$$

Finally, from (23) one derives $\mathbf{M}_{\text{eff}}$.

4. Numerical Example and Discussion

To illustrate the foregoing theory, let us calculate the effective macroscopic parameters of a symmetrical mixture where isotropic chiral inclusions are embedded in isotropic chiral environment. The material parameters of the components, relative to vacuum, are $\varepsilon_r = 2$, $\mu_r = 1.5$, and $\kappa = +1$ for the inclusions and $\varepsilon_r = 2$, $\mu_r = 1.5$, and $\kappa = -1$ for the background medium. In other words, the permittivity, permeability, and chirality parameters are the same for the inclusion and background; the only difference is the sign of κ. This means that the media are the same except that samples of inclusion medium and background medium are mirror images of each other: Inclusion medium is right-handed, whereas the background medium is left-handed.

With the formulas of the previous section the effective parameters of the mixture were calculated. The results are shown in Figures 1–3 as functions of the volume fraction of the inclusions. Four different mixing rules are treated: Maxwell Garnett ($\nu = 0$), $\nu = 1$, Bruggeman ($\nu = 2$), and Coherent Potential ($\nu = 3$). From the calculated curves, several striking conclusions can be drawn.

First, Figure 1 shows that although both components have the same electric permittivity, the mixture permittivity ε_{eff} is not the same. It is lower than that of the components but of course approaches that for the limiting cases $f = 0$ (no inclusions but homogeneous background) and $f = 1$ (no background, everything just inclusions). The effect on permittivity is the magnetoelectric coupling. If the chirality of the components vanished, the effective permittivity of the mixture would be constant, that of background and inclusion, independent of the volume fraction. The curves for $\nu = 1$ and $\nu = 2$ are symmetrical and 50/50 mixture yields the minimum permittivity. Maxwell

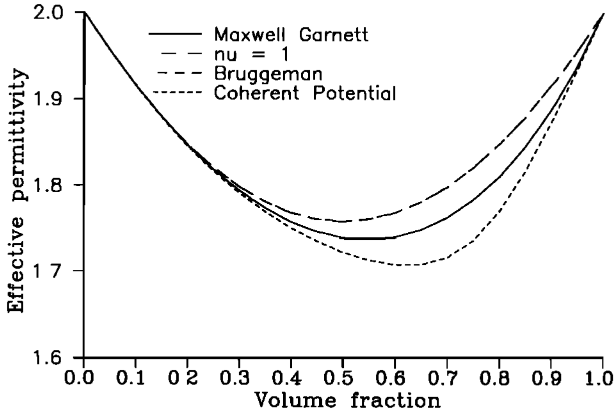

Figure 1. Macroscopic permittivity $\varepsilon_{\text{eff,r}}$ of a chiral-in-chiral isotropic mixture where the background is left-handed material with parameters $\varepsilon_r = 2$, $\mu_r = 1.5$, and $\kappa_r = -1$ and parameters $\varepsilon_r = 2$, $\mu_r = 1.5$, and $\kappa_r = +1$ for the inclusions (parameters given relative to vacuum). The variable f is the volume fraction of inclusions, and the four different models correspond to the values 0, 1, 2, and 3 for the parameter ν. Note that two of the curves (those corresponding to $\nu = 1$ and $\nu = 2$) are very similar.

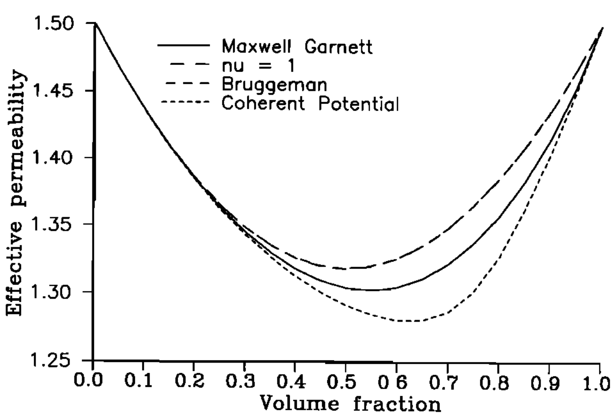

Figure 2. The same as Figure 1, except for macroscopic permeability $\mu_{\text{eff,r}}$.

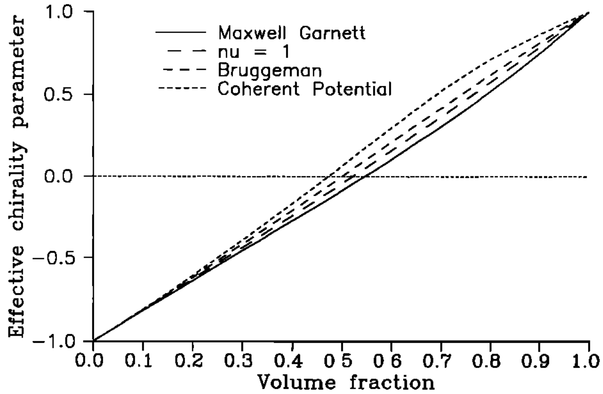

Figure 3. The same as Figure 1, except for macroscopic chirality parameter κ_{eff}.

Garnett and Coherent Potential are unsymmetrical, and the lowest effective permittivity of these four models is predicted by the Coherent Potential mixing rule.

Figure 2, the effective permeability μ_{eff}, looks similar in form to Figure 1. The different mixing rules behave similarly in terms of permeability as for permittivity. This is actually quite natural because of the duality between the electric and magnetic quantities. Chiral magnetoelectric coupling treats both electric and magnetic excitations on an equal footing.

Perhaps the most interesting information can be gleaned from Figure 3 which displays the effective chirality parameter of this mixture where right-handed inclusions occupy space in similar but left-handed ambient medium. Of course, all curves, independent of the mixing model, run from left-handed (negative chirality parameter) behavior to right-handed (positive chirality parameter) behavior as the volume fraction of the inclusions increases. But we can observe a difference in the predictions of the different models: An increase in the ν value increases the effective chirality parameter.

Also, the point where the effective chirality factor vanishes depends on the model. The existence of this crossing point $\kappa_{\text{eff}} = 0$ is natural: For a certain mixing ratio a left-right mixture appears racemic. We can observe that the Bruggeman model predicts racemization at exactly $f = 0.5$. For Coherent Potential this "zero-crossing" volume fraction is smaller, and for Maxwell Garnett it is larger. In the dielectric mixing studies the Bruggeman mixing rule is sometimes termed "Bruggeman symmetric mixing rule," and in light of Figure 3 this label is deserved: Out of these four models, Bruggeman indeed treats the inclusion and background most symmetrically.

This racemization prediction might serve as a sensitive test of the validity of mixing formulas. The sign of the effective chirality can be easily measured by sensing into which direction the polarization of a linearly polarized wave rotates as the wave propagates through a chiral sample [*Lindell et al.*, 1994]. For a racemic sample the rotation vanishes.

5. Conclusion

In conclusion, a systematical scheme has been presented to calculate effective material parameters of complex nonisotropic mixtures using an eigenvector decomposition of six-dyadic mixing formula. In the formula are contained several mixing rules that are established in the studies of dielectric mixtures. There are limitations to this formula (9). First, as all quasi-static mixing rules, this is also limited to low frequencies. It can be used for time-varying electromagnetic fields as long as the wavelength of the field is considerably larger than the size of the inclusion spheres. Secondly, the material has to be isotropic or biisotropic to which the polarizabilities are referred. And third, the (bi)anisotropic inclusion spheres have to be aligned so that their material dyadic has the same orientation in the global coordinate system. If there is an orientation distribution, the orientational averaging has to be included for the inclusion material matrix and the formula (9) needs to be modified.

Appendix

The most general biisotropic medium contains four scalar material parameters: In addition to the permittivity, permeability, and chirality parameters, the nonreciprocity parameter appears in the constitutive relations [*Lindell et al.*, 1994]. In the most recent literature the possible existence of nonreciprocal biisotropic media (sometimes called Tellegen media) has been doubted using arguments on the structure of the electromagnetic theory [*Lakhtakia and Weiglhofer*, 1994]. There is no broad agreement about this question [*Sihvola*, 1995].

In this appendix, however, we give general expressions for $\mathbf{M}_r$ for a general biisotropic background with $\mathbf{M}_0$ given by

$$\mathbf{M}_0 = \begin{bmatrix} \tilde{\varepsilon}_r \bar{\bar{I}} & (\tilde{\chi} - j\tilde{\kappa})\bar{\bar{I}} \\ (\tilde{\chi} + j\tilde{\kappa})\bar{\bar{I}} & \tilde{\mu}_r \bar{\bar{I}} \end{bmatrix} \tag{44}$$

The eigenvalues are

$$\lambda_{0,\pm} = \frac{\tilde{\varepsilon}_r + \tilde{\mu}_r}{2} \pm \sqrt{\left(\frac{\tilde{\varepsilon}_r - \tilde{\mu}_r}{2}\right)^2 + \tilde{\chi}^2 + \tilde{\kappa}^2} \tag{45}$$

which leads to

$$\sqrt{\mathbf{M}_0}^{-1} = \begin{bmatrix} \dfrac{1}{\sqrt{\hat{\varepsilon}_r}} \bar{\bar{I}} & \left(\dfrac{1}{\sqrt{\hat{\chi}}} - j \dfrac{1}{\sqrt{\hat{\kappa}}} \right) \bar{\bar{I}} \\ \left(\dfrac{1}{\sqrt{\hat{\chi}}} + j \dfrac{1}{\sqrt{\hat{\kappa}}} \right) \bar{\bar{I}} & \dfrac{1}{\sqrt{\hat{\mu}_r}} \bar{\bar{I}} \end{bmatrix} \tag{46}$$

where (32)–(34) remain valid and where

$$\frac{1}{\sqrt{\hat{\chi}}} = \frac{-\tilde{\chi}}{\sqrt{\lambda_{0,+}\lambda_{0,-}}\left(\sqrt{\lambda_{0,-}} + \sqrt{\lambda_{0,+}}\right)} \tag{47}$$

The relative matrix $\mathbf{M}_r$ derived from the general form (7) now becomes

$$\mathbf{M}_r = \begin{bmatrix} \check{\bar{\bar{\varepsilon}}}_r & c\check{\bar{\bar{\xi}}} \\ c\check{\bar{\bar{\zeta}}} & \check{\bar{\bar{\mu}}}_r \end{bmatrix} \tag{48}$$

where again $c = 1/\sqrt{\mu_0 \varepsilon_0}$ and

$$\check{\bar{\bar{\varepsilon}}}_r = \frac{\bar{\bar{\varepsilon}}_r}{\hat{\varepsilon}_r} + \frac{c\bar{\bar{\zeta}}}{\sqrt{\hat{\varepsilon}_r}}\left(\frac{1}{\sqrt{\hat{\chi}}} - \frac{j}{\sqrt{\hat{\kappa}}}\right) + \frac{c\bar{\bar{\xi}}}{\sqrt{\hat{\varepsilon}_r}}\left(\frac{1}{\sqrt{\hat{\chi}}} + \frac{j}{\sqrt{\hat{\kappa}}}\right) + \bar{\bar{\mu}}_r\left(\frac{1}{\hat{\chi}} + \frac{1}{\hat{\kappa}}\right) \tag{49}$$

$$\check{\bar{\bar{\mu}}}_r = \frac{\bar{\bar{\mu}}_r}{\hat{\mu}_r} + \frac{c\bar{\bar{\zeta}}}{\sqrt{\hat{\mu}_r}}\left(\frac{1}{\sqrt{\hat{\chi}}} - \frac{j}{\sqrt{\hat{\kappa}}}\right) + \frac{c\bar{\bar{\xi}}}{\sqrt{\hat{\mu}_r}}\left(\frac{1}{\sqrt{\hat{\chi}}} + \frac{j}{\sqrt{\hat{\kappa}}}\right) + \bar{\bar{\varepsilon}}_r\left(\frac{1}{\hat{\chi}} + \frac{1}{\hat{\kappa}}\right) \tag{50}$$

$$c\check{\bar{\bar{\zeta}}} = \left(\frac{\bar{\bar{\varepsilon}}_r}{\sqrt{\hat{\varepsilon}_r}} + \frac{\bar{\bar{\mu}}_r}{\sqrt{\hat{\mu}_r}}\right)\left(\frac{1}{\sqrt{\hat{\chi}}} + \frac{j}{\sqrt{\hat{\kappa}}}\right) + \frac{c\bar{\bar{\zeta}}}{\sqrt{\hat{\mu}_r \hat{\varepsilon}_r}} + c\bar{\bar{\xi}}\left(\frac{1}{\sqrt{\hat{\chi}}} + \frac{j}{\sqrt{\hat{\kappa}}}\right)^2 \tag{51}$$

$$c\check{\bar{\bar{\xi}}} = \left(\frac{\bar{\bar{\varepsilon}}_r}{\sqrt{\hat{\varepsilon}_r}} + \frac{\bar{\bar{\mu}}_r}{\sqrt{\hat{\mu}_r}}\right)\left(\frac{1}{\sqrt{\hat{\chi}}} - \frac{j}{\sqrt{\hat{\kappa}}}\right) + \frac{c\bar{\bar{\xi}}}{\sqrt{\hat{\mu}_r \hat{\varepsilon}_r}} + c\bar{\bar{\zeta}}\left(\frac{1}{\sqrt{\hat{\chi}}} - \frac{j}{\sqrt{\hat{\kappa}}}\right)^2 \tag{52}$$

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References

Böttcher, C. J. F., *Theory of Electric Polarization*, Elsevier, New York, 1952.

Bruggeman, D. A. G., Berechnung verschiedener physikalischer Konstanten von heterogenen Substanzen, I, Dielektrizitätskonstanten und Leitfähigkeiten der Mischkörper aus isotropen Substanzen, *Ann. Phys.*, *24*, 636–664, 1935.

Kampia, R. D., and A. Lakhtakia, Bruggeman model for chiral particulate composites, *J. Phys. D Appl. Phys.*, *25*, 1390–1394, 1992.

Kohler, W. E., and G. C. Papanicolaou, Some applications of the coherent potential approximation, in *Multiple Scattering and Waves in Random Media*, edited by P. L. Chow, W. E. Kohler, and G. C. Papanicolaou, pp. 199–223, North-Holland, New York, 1981.

Lakhtakia, A., and W. S. Weiglhofer, Are linear, nonreciprocal, biisotropic media forbidden? *IEEE Trans. Microwave Theory Tech.*, *42*(9), 1715–1716, 1994.

Lindell, I. V., A. H. Sihvola, S. A. Tretyakov, and A. J. Viitanen, *Electromagnetic Waves in Chiral and Bi-isotropic Media*, Artech, Boston, Mass., 1994.

Lindell, I. V., A. H. Sihvola, and K. Suchy, Six-vector formalism in electromagnetics of bianisotropic media, *J. Electromagn. Waves Appl.*, *9*(7/8), 887–903, 1995.

Maxwell Garnett, J. C., Colours in metal glasses and metal films, *Philos. Trans. R. Soc. London, Sect. A*, *3*, 385–420, 1904.

Rikte, S., Reconstruction of bi-isotropic material parameters using transient electromagnetic fields, *Tech. Rep. LUTEDX/(TEAT-7033)/1-22*, Lund Inst. of Tech., Lund, Sweden, 1994.

Sihvola, A., Self-consistency aspects of dielectric mixing theories, *IEEE Trans. Geosci. Remote Sens.*, *27*(4), 403–415, 1989.

Sihvola, A. H., Are nonreciprocal, bi-isotropic media forbidden indeed? *IEEE Trans. Microwave Theory Tech.*, *43*(9), 2160–2162, 1995. (See also *IEEE Trans. Microwave Theory Tech.*, *43*(12), 2722–2724, 1995.)

Sihvola, A. H., and O. P. M. Pekonen, Effective medium formulas for bi-anisotropic mixtures, *J. Phys. D Appl. Phys.*, *29*, 514–521, 1996. (Comment and Reply, *J. Phys. D Appl. Phys.*, in press, 1996.)

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